APPROXIMATION OF INHOMOGENEOUS THERMAL LINES VIA SERIES CONNECTION OF HOMOGENEOUS THERMAL LINES

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UDC 536.21.01

A method is given for approximating an arbitrary inhomogeneous thermal line (ITL) as a series connection of homogeneous thermal lines (HTL). The method is compared with other approximation methods.

Thermal objects have distributed parameters; they can be treated via the theory of circuits with distributed parameters, in particular via the A-parameter method [1].

By inhomogeneous thermal line one means a class of thermal objects for which the thermal conduction may be represented in one-dimensional form, i.e., the heat propagates along one of the coordinate axes (rectilinear or curvilinear), with the cross-sectional area, σ, the specific heat c_p, and thermal conductivity λ as continuous functions of coordinate.

Classical thermal objects such as an unbounded hollow cylinder and a hollow sphere may be considered as inhomogeneous thermal lines because their cross-sectional areas vary in the direction of heat propagation.

Other such lines are rods and thin symmetrical shells with constant or variable cross-sectional areas along the heat propagation direction, and with or without insulation (thermal loss).

The class of conduction problems that can be discussed for simple and composite ITL may be expanded by specifying the distributed heat sources (independent of temperature or directly proportional to temperature), and these may be functions of coordinates or time.

It is convenient to use the A parameters in the Laplace transforms for the solutions for this class of ITL for classical boundary conditions and for mixed ones; these can be obtained for a given ITL as a uniformly convergent series in powers of the Laplace transformation constant s.

The A parameters for a given ITL allow one to derive any system function (transfer function, input impedance, and so on), and thus one can describe the dynamic behavior in the time and frequency domains.

Closed forms are available for the A parameters for particular classes of ITL, especially in terms of special functions: hyperbolic, modified Bessel functions of the first and second kinds, and so on.

Here we describe a method of approximating an ITL via series-connected HTL. As a rule, only a few HTL are needed to provide a good approximation, so this method can be recommended for examining the behavior of ITL in the time and frequency domains. The method has definite advantages over approximating the ITL as series-connected elements with lumped parameters, and also over other methods of approximation: the first zeros in the A parameters and truncated series in s for the A parameters.

We consider an arbitrary ITL whose thermophysical parameters λ and c_p are functions of the coordinate r, while the cross-sectional area is some function of that coordinate: $\sigma = \sigma_0\sigma(r)$.

The behavior of the ITL is defined via two generalized fitting parameters [2]: the lengthwise thermal resistance

Institute of Heat and Mass Transfer, Academy of Sciences of the Belorussian SSR, Minsk. Translated from Inzhenerno-Fizikcheskii Zhurnal, Vol. 27, No. 4, pp. 732-737, October, 1974. Original article submitted November 21, 1973.

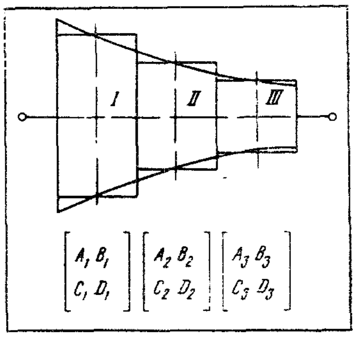

Fig. 1

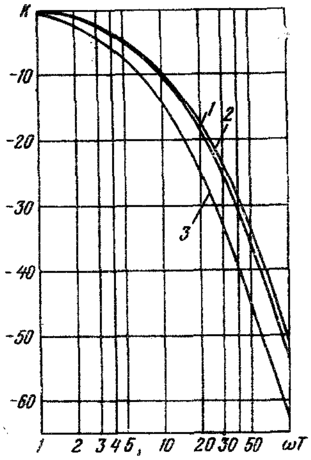

Fig. 2

Fig. 1. Representation of an inhomogeneous thermal line as series-connected homogeneous lines.

Fig. 2. Temperature transfer function K(αB) as a function of dimensionless frequency ωT for: 1) an inhomogeneous exponential thermal line insulated at the end; 2) the same for approximation via three homogeneous segments; 3) the same for approximation by three links with lumped parameters.

$$R_l(r) = R_{l0} f(r) = \frac{R_{l0}}{\lambda(r)\,\sigma(r)} \text{ [deg/W·m]} \tag{1}$$

and the lengthwise specific heat:

$$c_l(r) = c_{l0} g(r) = c_{l0} c_p(r)\,\sigma(r) \text{ [J·deg·m]} \tag{2}$$

where $R_{l0} = 1/(\lambda_0\sigma_0)$ and $c_{l0} = c_{p0}\sigma_0$ are the values, respectively, of the two functions at the start of the ITL ($r = 0$), while $f(r)$ and $g(r)$ are positive functions bounded in magnitude and having first derivatives continuous in the interval [0, 1].

First we demonstrate how to transfer from the given ITL to an analogous ITL in which only the cross-sectional area varies along the coordinate, while the thermophysical parameters are constant. We transform the r coordinate via a law $\xi(r)$ such as to get an ITL whose A parameters correspond to those of the initial line, but the lengthwise parameters of the new (analogous) line in terms of ξ take the form

$$\begin{aligned} R_l(\xi) &= R_{l0}/\sigma_{ef}(\xi); \\ c_l(\xi) &= c_{l0}\sigma_{ef}(\xi), \end{aligned} \quad 0 < \xi < \xi(l) = L, \tag{3}$$

where $\sigma_{ef}(\xi) = \sqrt{g/f}$, i.e., (3) gives the parameters of the analogous ITL inhomogeneous solely on account of the variation in σ_{ef}.

The required method of coordinate transformation is available [2], and takes the form:

$$\xi(r) = \int_0^r \sqrt{f(r)\,g(r)}\,dr = \int_0^r \sqrt{c_p(r)/\lambda(r)}\,dr. \tag{4}$$

The conduction problem is treated for this analogous ITL and thus gives the solution for the initial ITL; methods for transferring from ξ to r have also been given [2].

This shows why we restrict consideration to an ITL inhomogeneous solely in the area of cross-section $\sigma(r)$.

The following methods of approximation have previously been published: firstly, the ITL is represented as series-connected elements with lumped parameters, which is sometimes called the electrothermal analog method [3]; secondly, one uses only the first poles in the system function for the ITL; and thirdly, one can use a truncated Laplace-constant power series for the system function and locate two or three approximate poles [4].

TABLE 1. Parameter Comparison for an Inhomogeneous Exponential Thermal Line and Three Approximations

Coefficients and zeros of A parameters	Exact values (ITL)	Values from approximations		
		5 HTL	3 HTL	3 links with lumped parameters
		Coefficients in expansion of A as powers of s		
a_1	0,368	0,373	0,383	0,532
a_2	0,0285	0,0296	0,031	0,0531
a_3	0,000999	0,00096	0,0010	0,0014
		First zeros in A		
s_1	3,62	3,57	3,50	2,435
s_2	23,4	23,03	22,21	13,18
s_3	62,9	61,69	56,06	22,12
s_4	122,0	118,99	127,60	—
s_5	201,0	190,93	199,86	—
s_6	—	309,69	288,27	—
s_7	—	420,46	429,34	—
s_8	—	555,17	555,17	—
s_9	—	708,42	697,13	—
s_{10}	—	871,84	908,31	—

The first method is the commonest, and good computer programs are available, but some 40 sections are needed to obtain a good approximation [4].

In the second method, there is a considerable initial difficulty in defining the first poles in the system function.

The third method is simple and allows one to avoid using a computer, but it does not provide approximation with any desired degree of accuracy, and the error of approximation cannot be estimated.

The largest discrepancies from the exact transient response occur in all three methods at small times (high frequencies), because the system function has an infinite number of zeros and poles, but in all three methods it is replaced by a rational fraction with comparatively low-order polynomials in the numerator and denominator.

The approximation should be performed as follows.

As an example we consider an inhomogeneous exponential line with the characteristic

$$R_l = R_{l'} \exp(r); \quad c_l = c_{l'} \exp(-r).$$

To simplify the calculations we have assumed that $\sigma_0 = 1$, $l = 1$, $R_0 c_{p0} = 1$.

Figure 1 shows the ITL schematically, and the method of division into HTL segments of equal length, with the constant cross-section of each HTL taken as the cross-section of the initial ITL at the midpoint of the HTL.

For instance, if the ITL is split into n segments of identical length $d = l/n$, the corresponding areas of the HTL will be

$$\sigma_{\rm I} = \sigma\left(\frac{l}{2n}\right); \quad \sigma_{\rm II} = \sigma\left(\frac{3l}{2n}\right); \ \ldots; \ \sigma_n = \sigma\left(\frac{2n-1}{2n}\, l\right).$$

The A-parameter matrix for an HTL of given length d represented by the constant parameters R_0, c_{p0}, and σ_i = const is:

$$[A]_i = \begin{bmatrix} A_i & B_i \\ C_i & D_i \end{bmatrix} = \begin{bmatrix} \operatorname{ch} \sqrt{Ts} & \frac{Z_c}{\sigma_i} \operatorname{sh} \sqrt{Ts} \\ \frac{\sigma_i}{Z_c} \operatorname{sh} \sqrt{Ts} & \operatorname{ch} \sqrt{Ts} \end{bmatrix}, \tag{5}$$

where $T = R_0 c_{p0} d^2$, $Z_c = \sqrt{R_0/(s c_{p0})}$, $R_0 = 1/\lambda_0$.

The A-parameter matrix for a system of n series-connected thermal-line segments each with its A-parameter matrix $[A]_i$ is obtained by multiplying the latter, i.e., $[A] = [A]_{\rm I} [A]_{\rm II} \cdots [A]_n$.

For instance, for the A parameters of a system of three series-connected ITL we have

$$A = A_1A_2A_3 + B_1C_2A_3 + A_1B_2C_3 + B_1D_2C_3. \quad (6)$$

Equation (6) gives the initial equation for the A-parameters of a system of three HTL segments. Here we have

$$A = n_1 \operatorname{ch}^3\theta - n_2 \operatorname{ch}\theta \quad (7)$$

or

$$A = m_1 \operatorname{ch} 3\theta + m_2 \operatorname{ch}\theta, \quad (8)$$

where n_1, n_2, m_1, and m_2 are numerical coefficients and $\theta = \sqrt{Ts} = \sqrt{R_0 s c_{p0}}\,(l/3)$.

If the ITL is split into 5 HTL segments, the A-parameter is found as

$$A = n_1 \operatorname{ch}^5\theta - n_2 \operatorname{ch}^3\theta + n_3 \operatorname{ch}\theta \quad (9)$$

or

$$A = m_1 \operatorname{ch} 5\theta + m_2 \operatorname{ch} 3\theta + m_3 \operatorname{ch}\theta, \quad (10)$$

where n_1, n_2, n_3, m_1, m_2, and m_3 are numerical coefficients and $\theta = \sqrt{Ts} = \sqrt{R_0 s c_{p0}}\,(l/5)$.

The real negative zeros of the A-parameters in (8)-(10) and (7)-(9) may be determined graphically or analytically.

Table 1 gives the first zeros for the A-parameters and also the first three coefficients for the A-parameter of the inhomogeneous exponential thermal line, as well as the A-parameters derived by approximation via three and five HTL segments. These zeros were used to construct the frequency response of a system thermally insulated at the end (Fig. 2).

NOTATION

r and ξ, coordinates of inhomogeneous thermal line, m; l and L, total length of line, m; σ, cross-sectional area, m^2; λ, thermal conductivity, W/m·deg; c_p, specific heat, J/m^3·deg; R_1, linear thermal resistance, deg/W · m; c_l linear thermal capacity, J/deg · m; s, Laplace transformation constant.

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